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# Recent Progress on Pre-Hilbert-Space Logics and Their Measure Spaces[1]

**E. Chetcuti[2,3] and A. Dvurečenskij[2]**

The aim of this survey paper is to summarize the most recent results on pre-Hilbert-space logics and their corresponding measure spaces.

**KEY WORDS:** Inner product space (pre-Hilbert space); orthogonally closed subspace; splitting subspace; quasi-splitting subspace; charge 2003.

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## 1. INTRODUCTION

One of the most important models used in the foundation of quantum mechanics is the projection lattice of a Hilbert space (Birkhoff and von Neumann, 1936) (see also Varadarajan, 1985; Kalmbach, 1983, 1986; Pták and Pulmannová, 1991, and others), i.e. the collection $L(H)$ of all closed subspaces of a Hilbert space $H$, endowed with the inclusion relation $\subseteq$ and the orthocomplementation $\perp$. The absence of distributivity in $L(H)$ gives rise to Heisenberg's uncertainty principle. As it was observed by Husimi (1937), $L(H)$ always satisfies the *orthomodular law*:

$$\text{if } A, B \in L(H) \quad \text{and} \quad A \subseteq B, \quad \text{then} \quad B = A \vee (B \wedge A^{\perp}).$$

The key axiom in Dirac's formulation of quantum mechanics is the superposition principle, which is connected with the linear structure of the Hilbert space. It expresses the circumstance that a normalized linear combination of two unit vectors (i.e. superposition of the states corresponding to these vectors) also represents a pure state. This suggests one to consider general linear spaces that are endowed with a bilinear form instead of Hilbert spaces only. If we assume that the bilinear form is strictly positive, then our space is a quadratic space, in particular, if the

[1] The paper is dedicated to the memory of Prof. Günter Bruns.

[2] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 74 Bratislava, Slovakia.

[3] To whom correspondence should be addressed at Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 74 Bratislava, Slovakia; e-mail: chetcuti@mat.savba.sk.

linear structure is assumed to be over the field of real or complex numbers, or the division ring of quaternions, we get an inner product space (pre-Hilbert space). Put in another way, a natural question can be put regarding the role of the purely mathematical property of (topological) completeness of $H$ with respect to the physical interpretation. A lot of work has been done in this line. Among the most inspiring contributions, one would mention (Amemiya and Araki, 1966–1967; Gross and Keller, 1977; Holland, 1969). On the other hand, Keller (1980) found a surprising example of a nonarchimedean orthomodular quadratic space such that the corresponding projection logic has similar properties to $L(H)$. Another very surprising result is due to Solèr who showed that if an orthomodular quadratic space admits at least one orthonormal sequence, then this space has to be a Hilbert space (Solèr, 1995). Apart from algebraic properties, measure-theoretic properties of pre-Hilbert-space logics are also studied. The first contribution in this line is the result of Hamhalter and Pták (1987). At the beginning of 1990s some results in this area were collected by Dvurečenskij in the book (Dvurečenskij, 1992) and the article (Dvurečenskij, 1993). These results were also presented at the first IQSA meeting (1992) in Castiglioncello, Italy. Recently, Hamhalter (2003) gathered results known in the last period. The aim of this survey paper is to summarize the most recent results on pre-Hilbert-space logics and their corresponding measure spaces.

## 2. PRE-HILBERT-SPACE LOGICS

Let $S$ be an inner product space. Unless otherwise stated we shall not assume that $S$ is (metrically) complete. On the other hand, we assume that the linear structure of $S$ is defined over either the field of real or complex numbers, or the division ring of quaternions. For any subspace $M$ of $S$ we shall write $\overline{M}$ for the completion of $M$ and we shall denote by $[x]$ the one-dimensional subspace of $S$ generated by the non-zero vector $x$. We denote by $\langle\cdot,\cdot\rangle$, the inner product on $S$. For any subset $A$ of $S$ we denote by $A^\perp$ the orthogonal complement of $A$, i.e. $A^\perp = \{x \in S : \langle x, y\rangle = 0 \text{ for all } y \in A\}$. If $S$ is a complete inner product space (i.e. a Hilbert space), then every closed subset is complete. This means that the set of closed subspaces coincides with the set of complete subspaces. This is not the case in general. When we drop the assumption of completeness on $S$, we get a spectrum of families of closed subspaces of $S$. Here we consider five of them.

$$P(S) = \{M \subseteq S : \dim M < \infty \text{ or } M = A^\perp, A \subset S, \dim A < \infty\}$$

$$C(S) = \{M \subseteq S : M \text{ is complete or concomplete}\}$$

$$E(S) = \{M \subseteq S : S = M \oplus M^\perp\}$$

$$E_q(S) = \{M \subseteq S : M \text{ is closed in } S \text{ and } M \oplus M^\perp \text{ is dense in } S\}$$

$$F(S) = \{M \subseteq S : M = M^{\perp\perp}\}$$

A subspace $M$ of $S$ is cocomplete if there exists a complete subspace $A$ of $S$ such that $M = A^{\perp}$. For any inner product space $S$ we have

$$P(S) \subseteq C(S) \subseteq E(S) \subseteq E_q(S) \subseteq F(S).$$

The family $C(S)$ was first introduced in Pták and Weber (2001). We say that $M$ is *orthogonally closed* if $M \in F(S)$. An element of $E(S)$ is a said to be a *splitting subspace* of $S$. The family $E_q(S)$ was recently introduced and studied in Buhagiar and Chetcuti (in press). A subspace $M \in E_q(S)$ is said to be a *quasi-splitting subspace* of $S$.

When endowed with the partial ordering of set-inclusion $\subseteq$ and orthocomplementation $\perp$, the families defined above carry an algebraic structure with orthocomplementation. When $S$ is not complete, the algebraic structures of these families differ considerably. If $S$ is complete, then $C(S) = F(S)$. In contrast, if $S$ is an infinite-dimensional inner product space with a countable linear dimension, then

$$C(S) \subsetneq E(S) \subsetneq E_q(S) \subsetneq F(S).$$

We further note that when $S$ is a hyperplane in $\overline{S}$, then $C(S) = E(S)$ and $E_q(S) = F(S)$ (refer to Pták and Weber, 2001; Buhagiar and Chetcuti, in press). From the result of Amemiya and Araki it follows that $S$ is complete if, and only if, $E(S) = F(S)$.

The family $F(S)$ of orthogonally closed subspaces is a complete lattice, where for the meet and join we have

$$\bigwedge_{i\in I} M_{\mathrm{i}} = \bigcap_{i\in I} M_{\mathrm{i}} \quad \text{and} \quad \bigvee_{i\in I} M_{\mathrm{i}} = \left\{ \mathrm{span}\{\bigcup_{i\in I} M_{\mathrm{i}}\} \right\}^{\perp\perp},$$

respectively. Observe that the family $F(S)$ is the largest family of subspaces that can be identified with events of a quantum system—the condition $M = M^{\perp\perp}$ is the bare minimum. Moreover, Amemiya–Araki theorem asserts that $F(S)$ cannot enjoy the orthomodular property unless $F(S) = E(S)$ (i.e. $S$ is a Hilbert space).

Let us now turn to the algebraic structure of $E(S)$ and $C(S)$. From the definition of $E(S)$ it is immediately seen that for any $A, B \in E(S)$ such that $A \subseteq B$, we have $B = A \oplus (A^{\perp} \cap B)$. Since $A^{\perp} \cap B \in E(S)$, it follows that $E(S)$ is an orthomodular poset. The same holds for $C(S)$. However, unless $S$ is a Hilbert space, neither $C(S)$, nor $E(S)$, can be complete lattices. Gross and Keller (1977) proved that $S$ is complete if, and only if, $E(S)$ is a complete lattice. This was subsequently and successively strengthened as follows: $S$ is complete if, and only if, (i) $E(S)$ is a $\sigma$-lattice (Cattaneo and Marino, 1986), (ii) $E(S)$ is a $\sigma$-orthomodular poset (Dvurečenskij, 1988), and (iii) $E(S)$ is atomically weakly $\sigma$-complete (Pták and Weber, 1998). (Since the atoms of $E(S)$ are the one-dimensional subspaces of $S$, it follows immediately that $S$ is complete if, and only if, $C(S)$ is atomically weakly

$\sigma$-complete.) In addition, in Dvurečenskij (2001) it was further shown that $S$ is complete if, and only if, $E(S)$ satisfies the $\sigma$-Riesz interpolation property.

A natural question arising in this connection was that of whether $E(S)$ could be a lattice for $S$ incomplete. This question was answered in Pták and Weber (2001), by careful analysis of the lattice properties of $E(S)$ and $C(S)$. A hyperplane $S$ of $\overline{S}$ was constructed such that $E(S) = P(S)$, and thus $E(S)(= C(S))$ is a modular lattice. However, in Gross and Keller (1977) it was shown that for any inner product space having a countable (infinite) linear dimension, $E(S)$ is not a lattice. Thus, the lattice properties of $E(S)$ do not seem to have an explicit bearing on the metric completeness of $S$.

The family $E_q(S)$ was introduced as an intermediate family between $E(S)$ and $F(S)$. We recall that $E(S)$ can be very "poor" (sometimes $E(S) = P(S)$, i.e. $E(S)$ might not contain any Boolean $\sigma$-subalgebras), whereas $F(S)$ is always "rich" (it always has a vast supply of Boolean $\sigma$-subalgebras). Whereas the members of $E_q(S)$ should behave somewhat similarly to the splitting subspaces, $E_q(S)$ (at least when $S$ is assumed to be separable) is always furnished with a "good supply" of Boolean $\sigma$-subalgebras. Indeed, by the Gram–Schmidt orthonormalization procedure we can convert any countable dense subset of $S$ into an orthonormal basis $\{x_i\}$. Clearly, the image of the map

$$\Gamma : (I \in 2^{\mathbb{N}}) \mapsto \left(S \bigcap \overline{\mathrm{span}}\{x_i : i \in I\} \in E_q(S)\right)$$

is a Boolean $\sigma$-subalgebra of $E_q(S)$. The lattice structure of $E_q(S)$ turns out to be strongly dependent on the particular choice of the inner product space itself. If $S$ is a hyperplane in $\overline{S}$, then $E_q(S) = F(S)$ and thus $E_q(S)$ is a complete lattice. This is in contrast to the situation with $E(S)$, where such a rich structure is allowed only in the case when $S$ is complete. Yet, if $S$ has a countable linear dimension, then $E_q(S)$ is not a lattice even. Buhagiar and Chetcuti (in press) put as a conjecture the following.

**Conjecture 2.1.** $E_q(S) = E(S)$ *if, and only if, $S$ is complete.*

This was not completely proved. It was done only for the case when $\dim \overline{S}/S < \infty$.

**Theorem 2.2.** *Let $S$ be an inner product space such that* $\dim \overline{S}/S < \infty$. *Suppose that $E_q(S) = E(S)$. Then $S = \overline{S}$.*

Despite from the fact that not every inner product space possesses an orthonormal basis, it is still possible (and useful) to define the orthogonal dimension of an inner product space as the cardinality of any maximal orthonormal system in $S$. However, in contrast to Hilbert spaces, this orthogonal dimension reveals very little information about the properties of the space itself. We recall the well-known fact that two Hilbert spaces are isomorphic if, and only if, they have the same orthogonal dimension. This is not the case with incomplete inner product spaces, as the example given by Gudder (1975) exhibits. Unless we assume completeness, isomorphism between two inner product spaces can only be proved by exhibiting a bijective unitary transformation between the two spaces.

In Buhagiar and Chetcuti (2004), the following question was analyzed: Suppose that we are given two separable inner product spaces $S_1$ and $S_2$ over $\mathbb{R}$, such that $P(S_1)$ is algebraically equivalent to $P(S_2)$ as modular lattices. What can be said about $S_1$ and $S_2$? Using Gleason theorem, it was proved that, in such case, $S_1$ and $S_2$ are unitarily equivalent, i.e. there exists a bijective unitary transformation between $S_1$ and $S_2$. This was then extended by Pulmannová (2005). Her proof is based on results in projective geometries. One can look at this as a generalization of the classical Wigner's (1959) theorem.

**Theorem 2.3.** *Let $S_1$ and $S_2$ be two inner product spaces over the same field. Then the following statements are equivalent:*

*(i)* *$S_1$ is isomorphic (or anti-isomorphic) to $S_2$ (as inner product spaces).*
*(ii)* *$P(S_1)$ is isomorphic to $P(S_2)$ (as modular lattices).*
*(iii)* *$C(S_1)$ is isomorphic to $C(S_2)$ (as orthomodular posets).*
*(iv)* *$E(S_1)$ is isomorphic to $E(S_2)$ (as orthomodular posets).*
*(v)* *$F(S_1)$ is isomorphic to $F(S_2)$ (as complete lattices).*

## 3. MEASURES ON PRE-HILBERT-SPACE LOGICS

A *charge* on any of the respective families defined above is an additive signed—measure. Formally, if we let $\mathcal{L}$ be any of $P(S)$, $C(S)$, $E(S)$, $E_q(S)$ or $F(S)$, a charge $m$ on $\mathcal{L}$ is a map $m : \mathcal{L} \to \mathbb{R}$, such that

$$m(A \wedge B) = m(A) + m(B), \quad \text{whenever} \quad A, B \in \mathcal{L} \text{ and } A \perp B. \tag{3.1}$$

A charge $m$ is said to be:

(i) *completely-additive*, if Eq. (3.1) holds for any collection $\{A_i : i \in I\}$ of pairwise orthogonal subspaces in $\mathcal{L}$ such that the supremum $\bigvee_{i \in I} A_i$ exists in $\mathcal{L}$;
(ii) *$\sigma$-additive*, if Eq. (3.1) holds for any sequence $\{A_i : i \in \mathbb{N}\}$ of pairwise orthogonal subspaces in $\mathcal{L}$ such that the supremum $\bigvee_{i \in \mathbb{N}} A_i$ exists in $\mathcal{L}$;
(iii) *bounded*, if there exists $k > 0$ such that $|m(A)| \leq k$ for each $A$ in $\mathcal{L}$;
(iv) *regular*, if for every $A$ in $\mathcal{L}$, and every $\epsilon > 0$, there exists a finite-dimensional subspace $M$, contained in $A$, such that $|m(A) - m(M)| \leq \epsilon$;
(v) *free*, if $m(A)$ is zero for every finite-dimensional subspace $A$ in $\mathcal{L}$.

A *state* $s$ on $\mathcal{L}$ is a normalized (i.e. $s(S) = 1$) positive charge. The set of all states on $\mathcal{L}$, denoted by $\mathcal{S}(\mathcal{L})$, is a convex subset of the cube $[0, 1]^{\mathcal{L}}$. When endowed with the product topology, $[0, 1]^{\mathcal{L}}$ is a compact space, and since $\mathcal{S}(\mathcal{L})$ is closed in $[0, 1]^{\mathcal{L}}$, it follows that the state space of any of the respective families described earlier, is a compact, convex topological space.

By solving the problem that was originally posed by Mackey, Gleason (Gleason, 1957) did not only succeed in describing all the $\sigma$-additive states on $L(H)$ (where $H$ is a separable Hilbert space)—he has also revealed the very delicate interplay that exists between the measure-theoretic properties of $L(H)$ and the geometric structure of $H$.

**Theorem 3.1.** (Gleason, 1957) *Let $H$ be a separable Hilbert space, dim $H \geq 3$. For any $\sigma$-additive state $s$ on $L(H)$, there exists a unique positive trace operator $T$, with unit trace, on $H$ such that*

$$s(M) = \operatorname{Tr} T P_M, \quad M \in L(H).$$

Gleason's theorem for bounded, $\sigma$-additive charges was independently proved by Sherstnev (1974) and Dvurečenskij (1978), and extended for non-separable Hilbert spaces by Eilers and Horst (1975), and Drisch (1979). It was later shown, by Dorofeev and Sherstnev (1990) that every completely additive charge on $L(H)$ ($\dim H = \infty$) is necessarily bounded. This is a very deep and surprising result that is not true in the case of finite-dimensional Hilbert spaces.

One of the main (and widely used) consequences of Gleason's theorem can be easily exhibited here. It is clear that the $\sigma$-additive states on $L(H)$ are in one-to-one correspondence with the frame functions on the unit sphere $\mathbb{S}(H)$ of $H$. (By a frame function on $\mathbb{S}(H)$ we understand a mapping $f : \mathbb{S}(H) \to [0, 1]$ such that for any orthonormal basis $\{x_i\}$ of $H$, we have $\sum_i f(x_i) = 1$.) In view of Gleason's theorem, for every frame function $f$ on $\mathbb{S}(H)$, there exists a unique positive trace operator $T$ (with unit trace) on $H$, such that $f(u) = \langle Tu, u\rangle$, for all $u \in \mathbb{S}(H)$. This means that

$$\begin{aligned} |f(u) - f(v)| &= |\langle Tu, u\rangle - \langle Tv, v\rangle| \\ &\leq |\langle Tu, u\rangle - \langle Tu, v\rangle| + |\langle Tu, v\rangle - \langle Tv, v\rangle| \\ &\leq 2\|T\| \cdot \|u - v\|, \end{aligned}$$

which implies that $f$ is uniformly continuous on $\mathbb{S}(H)$. Observe, that as an immediate consequence of this, we have that $L(H)$ ($\dim H \geq 3$) does not admit any two-valued state. (This is related to the problem of hidden variables.) We remark that Gleason's theorem is not true for the case when $H$ is two-dimensional—it is straightforward to check that $L(\mathbb{R}^2)$ admits plenty of two-valued states.

### 3.1. Measures on $F(S)$

The first sound result concerning the state space $\mathcal{S}(F(S))$ is due to Hamhalter and Pták (1987). They proved that a separable inner product space $S$ is complete if, and only if, $\mathcal{S}(F(S))$ contains a $\sigma$-additive state. Their proof consists mainly in showing that if $F(S)$ admits a $\sigma$-additive state, then the state space $\mathcal{S}(F(S))$ must separate $F(S)$, i.e. whenever $A, B \in F(S)$, such that $A \subsetneq B$, there must exist a state $s$ satisfying $s(A) < s(B)$. This, on the other hand, forces $F(S)$ to be orthomodular, which—in view of the Amemiya–Araki theorem—implies that $S$ must be complete. This result was generalized for non-separable inner product spaces and completely-additive charges—see Dvurečenskij (1992) for a complete survey.

A natural problem that erupted immediately after the publication of the Hamhalter–Pták result was: *Is the state space of $F(S)$ necessarily empty for incomplete inner product spaces* (see Pták, 1988; Dvurečenskij, 1992, Problem 4.3.12). The first contribution towards an answer to this question was due to Dvurečenskij *et al.* (1990); It was proved that an inner product space $S$ is complete if, and only if, $F(S)$ admits a regular state. Indeed, it was shown that every regular state on $F(S)$ is necessarily completely-additive.

The state-on-$F(S)$-problem (as it was often referred to) remained open for a considerably longer time. There were partial results that contributed to the better understanding of the nature of what might be a possible candidate for a state on $F(S)$, in the case that $S$ is not Hilbert. For instance, it was shown that $F(S)$ does not admit any two-valued states—see Dvurečenskij (1992). It was further shown, in Dvurečenskij and Pták (2002), that for any inner product space, such that $\dim S \geq 3$, if $\mathcal{S}(F(S)) \neq \emptyset$, then for any state $s$ on $F(S)$, we have Range$(s) = [0,1]$. This last result was later generalized in the paper in Chetcuti and Dvurečenskij (2003). A thorough investigation of the possible ranges of charges on $F(S)(\dim\ S \geq 3)$ was given. It was shown that, if the non-zero charge $m$ is bounded, then for infinite-dimensional inner product spaces, Range ($m$) is always convex. A counter-example is also given to show that the same need not be true if the charge is not bounded. Furthermore, the notion of *sign-preserving charges* was introduced.

*Definition 3.2.* A charge m on $F(S)$ is said to satisfy the sign-preserving property (or we say that m is a sign-preserving charge) if for any countable collection $\{N_i : i \in \mathbb{N}\}$ of orthogonal finite-dimensional subspaces in $F(S)$ satisfying $m(N_i) > 0$, (resp. $m(N_i) < 0$) for all $i \in \mathbb{N}$, it follows that $m(\bigvee_{i\in\mathbb{N}} N_i) \geq 0$, (resp. $m(\bigvee_{i\in\mathbb{N}} N_i) \leq 0$).

It was shown that the range of charges on $F(S)(\dim\ S = \infty)$ satisfying this property and the Jauch–Piron property is always convex.

In Chetcuti and Dvurečenskij (2005), sign-preserving charges on $F(S)$ were studied in connection with the completeness of the inner product space $S$. It was firstly shown that every sign-preserving charge is bounded on $P(S)$ whenever $\dim S = \infty$. (When $\dim S < \infty$, this need not be true.) Then, the following completeness criterion was proved.

**Theorem 3.3.** *An inner product space $S$ is complete if, and only if, $F(S)$ admits a (non-zero) sign-preserving, regular charge.*

Another contribution to the state-on-$F(S)$-problem was given in Chetcuti and Dvurečenskij (2003). In this paper, it was shown that for any incomplete inner product space $S$, the state space $\mathcal{S}(F(S))$ is very poor—it can only contain free states.

**Theorem 3.4.** *An inner product space $S$ is complete if, and only if, $F(S)$ admits a state that is not free.*

The argument of the proof for this theorem is different from that used by Hamhalter and Pták in their original paper cited earlier. Here, we used the fact that for any orthogonally closed subspace $A$ in $S$, each subspace $M$ of $A$ that is orthogonally closed relative to $A$, is necessarily orthogonally closed relative to $S$, i.e.

$$M \subset A \quad M^{\perp_A \perp_A} = M \quad \text{implies} \quad M^{\perp\perp} = M,$$

where $M^{\perp_A} = M^{\perp} \cap A$. In this proof, however, use was made of the fact that a state is positive, i.e. monotonic. In contrast to the other two criteria (Hamhalter and Pták, 1987; Dvurečenskij *et al.*, 1990), the proof does not extend directly to the case of bounded charges. It is still an open problem whether the state space $\mathcal{S}(F(S))$, for an incomplete $S$, can admit any bounded charges that are not free.

So, a candidate for what might be a state on $F(S)$, for an incomplete inner product space $S$, has to be rather "bizarre," in the sense that such a measure must vanish on all the finite-dimensional subspaces of $S$, and yet, must take all the values in the unit interval [0, 1]. Indeed, it was supposed that such a measure could hardly exist, and all efforts were directed towards an answer that, for incomplete $S$, excludes completely the possibility of having a state on $F(S)$. The state-on-$F(S)$-problem was finally resolved in Chetcuti and Dvurečenskij (2004). A dense hyperplane $S$ of a separable Hilbert space was exhibited for which $\mathcal{S}(F(S))$ is not empty. In addition, it was also shown that $\mathcal{S}(F(S))$ is non-empty when $S$ has a countable linear dimension.

So, although "relatively poor," the state space $\mathcal{S}(F(S))$ need not be empty for incomplete inner product spaces.

$\mathcal{S}(F(S))$ is fully described in the following theorem (see Chetcuti and Dvurečenskij, 2005) for a large class of incomplete inner product spaces—*strongly dense spaces*. An inner product space is said to be *strongly dense* (in its completion) if for every infinite-dimensional closed subspace $M$ of $\overline{S}$, we have $M \cap S \neq \emptyset$. Each inner product space $S$ satisfying $\dim \overline{S}/S < 2^{\aleph_0}$ is strongly dense. In particular, each dense hyperplane of a Hilbert space is strongly dense. For such spaces, $F(S)$ admits plenty of states, and moreover, each state on $F(S)$ is a restriction of some free state on $L(H)$ (where $H = \overline{S}$).

**Theorem 3.5.** *Let $S$ be an incomplete, strongly dense inner product space. There is an affine homeomorphism $\phi : s \mapsto s^{\phi}$ between the state space of $F(S)$ and the face of $\mathcal{S}(L(H))$ (where $H = \overline{S}$) consisting of the free states on $L(H)$. Each state $s$ on $F(S)$ is the restriction of $s^{\phi}$, i.e. $s(M) = s^{\phi}(\overline{M})$, for all $M \in F(S)$.*

In particular, when $S$ is over the complex field, we have the following corollary.

**Corollary 3.6.** *Let $S$ be an incomplete, strongly dense inner product space over the complex field. The state space of $F(S)$ is affinely homeomorphic to the state space of the Calkin algebra associated with the completion $\overline{S}$ of $S$.*

### 3.2. Measures on $E(S)$ and $C(S)$

In contrast to $F(S)$, the orthomodular posets $E(S)$ and $C(S)$ always allow for a separating system of charges. This is in view of the fact that $E(S)$ (and $C(S)$) can always be embedded (as orthomodular posets) in $L(\overline{S})$, and thus each charge on $L(\overline{S})$ induces a charge on $E(S)$ (and $C(S)$). (The embedding is $(M \in E(S)) \mapsto (\overline{M} \in L(\overline{S}))$.) However, in Dvurečenskij and Pulmannová (1989), it was shown that for an incomplete inner product space $S$, $E(S)$ does not allow for any completely additive charge. This implies that the embedding does not preserve infinite suprema. This is where the notion of regularity "acts as a substitute" to complete additivity. Caution is required however! Unless we impose some boundedness condition, the notion of regularity is strictly weaker than that of complete additivity.

For any Hermitian trace operator $T$ defined on $\overline{S}$, the map

$$m : (M \in E(S)) \mapsto (\mathrm{Tr}\,(T P_{\overline{M}}) \in \mathbb{R}). \tag{3.2}$$

defines a bounded regular charge on $E(S)$. In Chetcuti *et al.* (in press), it was shown that every bounded regular charge $m$ on $E(S)$ arises in this way. This means that the set of bounded regular charges on $E(S)$ is in one-to-one correspondence with the set of completely additive charges on $L(\overline{S})$. Moreover, if $S$ is complete, then the set of bounded regular charges on $E(S)$ coincides with the set of completely additive charges.

Not without a little bit of surprise, in Chetcuti and Dvurečenskij (2004), the following theorem was shown.

**Theorem 3.7.** *The set of completely additive charges on* $L(\overline{S})$, $\dim S = \infty$, *is a proper subset of the set of regular charges.*

A regular charge $m$ having its range contained in $\mathbb{Q}$ was exhibited. In view of the result of Chetcuti and Dvurečenskij (2003), it follows that $m$ is not bounded. Thus, $m$ is not completely additive, since every completely additive charge on $L(H)$ (where $H$ is an infinite-dimensional Hilbert space) is necessarily bounded (this is the result of Dorofeev and Sherstnev).

### 3.3. Convergence of Regular Charges

Now we consider convergence problems for charges on $E(S)$. Convergence theorems for completely additive charges on $E(S)$, in the case when $S$ is a Hilbert space, were originally studied by Jajte (1972). We say that a sequence $\{m_i : i \in \mathbb{N}\}$ of bounded regular charges on $E(S)$ is *uniformly regular* if for any $M \in E(S)$ and for any $\epsilon > 0$ there exists a finite-dimensional subspace $M_0$ of $M$ such that $|m_i(M) - m_i(M_0)| < \epsilon$ for each $m_i$. The following theorem is Nikodym convergence theorem in the $E(S)$-setup, where $S$ is a Hilbert space. This follows directly from Dvurečenskij (1992, Theorem 3.10.1).

**Theorem 3.8.** *Let $S$ be a Hilbert space and $\{m_i : i \in \mathbb{N}\}$ be a sequence of regular bounded charges on $L(S)$ converging pointwise on $E(S)$. Then the limit function is a bounded and regular charge. (This means that the set of bounded regular charges on $E(S)$ is weakly sequentially closed.) Moreover, the sequence $\{m_i : i \in \mathbb{N}\}$ is uniformly regular.*

In Chetcuti *et al.* (in press) the same problem was investigated for the case when $S$ is incomplete. It was shown that this case does not allow for such a clear answer. It was shown that if $S$ is the dense hyperplane (that was originally constructed by Pták and Weber (2001)) for which $E(S) = P(S)$, then the set of regular bounded charges on $E(S)$ is not weakly sequentially closed—not even if we restrict ourselves to states. This provides a negative answer to the problem posed by Dvurečenskij (see Dvurečenskij, 1992, Problem 4.3.15). The following theorem was proved in Chetcuti *et al.* (in press) and it gives a sufficient condition, under which the limit of a pointwise convergent sequence of bounded regular charges on $E(S)$ is regular.

**Theorem 3.9.** *Let $\{m_i : i \in \mathbb{N}\}$ be a sequence of bounded regular charges on $E(S)$ converging pointwise on $E(S)$. Suppose that there exists a bounded regular charge $m_0$ on $E(S)$ such that $|m_i([x])| \geq |m_0([x])|$ for each $i \in \mathbb{N}$ and for every unit vector $x$ of $S$. Then the limit function is a bounded regular charge on $E(S)$. Moreover, the sequence $\{m_i : i \in \mathbb{N}\}$ is uniformly regular.*

## 4. OPEN PROBLEMS

At the end of this survey article we put a list of natural open problems in this field. Some of the problems were already mentioned in the text, but we enlist them here for better reference.

1. Prove or disprove Conjecture 2.1.
2. Does there exist an inner product space $S$ such that $F(S)$ is stateless?
3. Is it possible to extend every free state on $E(S)$ to a state on $L(\overline{S})$? (See also Dvurečenskij, 1992, Problem 4.3.13, Proposition 4.3.14.) The case when $S$ is strongly dense seems to be most promising.
4. If $\dim \overline{S}/S = n$, the mapping $s : E(S) \to [0, 1]$ defined by $s(A) = \frac{\dim \overline{A}/A}{n}$, $A \in E(S)$ is a (Gleason-type) free state on $E(S)$ such that Range$(s) \subset \{0, 1/n, 2/n, \dots, 1\}$. Is it possible to characterize all (discrete) free states on $E(S)$ when $\dim \overline{S}/S = n$ (in a Gleason-type formula) for $S$ being strongly dense? (Compare with Theorem 3.5 for states on $F(S)$.)
5. Suppose that a sequence $\{m_i : i \in \mathbb{N}\}$ of regular bounded charges on $E(S)$ converges to some charge $m$. Is $m$ bounded? This is the case when the inner product space is Hilbert; but how is it in the general case?

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## REFERENCES

Amemiya, I. and Araki, H. (1966–1967). A remark on Piron's paper, *Publication of the Research Institute of Mathematics and Science, Kyoto University Series A* **2**, 423–427.

Birkhoff, G. and von Neumann, J. (1936). The logic of quantum mechanics, *Annals of Mathematics* **37**, 823–843.

Buhagiar, D. and Chetcuti, E. (2004). On isomorphisms of inner product spaces, *Math. Slovaca* **54**, 109–117.

Buhagiar, D. and Chetcuti, E. (in press). Quasi-splitting subspaces in a pre-Hilbert space, *Mathematische Nachrichten*.

Cattaneo, G. and Marino, G. (1986). Completeness ofinner product spaces with respect to splitting subspaces, *Letters on Mathematical Physics* **11**, 15–20.

Chetcuti, E. and Dvurečenskij, A. (2003). Range of charges on orthogonally closed subspaces of an inner product space, *International Journal of Theoretical Physics* **42**, 1927–1942.

Chetcuti, E. and Dvurečenskij, A. (2003). A finitely additive state criterion for the completeness of inner product spaces, *Letters on Mathematical Physics* **64**, 221–227.

Chetcuti, E. and Dvurečenskij, A. (2005). Boundedness of sign-preserving charges, regularity and the completeness of inner product spaces, *Journal of Australian Mathematical Society*, **78**, 199–210.

Chetcuti, E. and Dvurečenskij, A. (2004). The existence of finitely additive states on orthogonally closed sub-spaces of incomplete inner product spaces, *Letters on Mathematical Physics* **67**, 75–80.

Chetcuti, E. and Dvurečenskij, A. (2004). Measures on the splitting subspaces of an inner product space, *International Journal of Theoretical Physics* **43**, 369–384.

Chetcuti, E. and Dvurečenskij, A. (2005). The state-space of the lattice of orthogonally closed subspaces, *Glasgow Mathematical Journal* **47**, 213–220.

Chetcuti, E., de Lucia, P., and Dvurečenskij, A. (2006). Sequential convergence of regular measures on prehilbert space logics, *Journal of Mathematical Analysis and Applications*, in press.

Dorofeev, S. V. and Sherstnev, A. N. (1990). Frame-type functions and their applications, *Izv. vuzov matem.* (4), 23–29 (in Russian).

Drisch, T. (1979). Generalization of Gleason's theorem, *International Journal of Theoretical Physics* **18**, 239–243.

Dvurečenskij, A. (1978). Signed states on a logic, *Math. Slovaca* **28**, 33–40.

Dvurečenskij, A. (1988). Completeness of inner product spaces and quantum logic of splitting subspaces, *Letters on Mathematical Physics* **15**, 231–235.

Dvurečenskij, A. (1992). *Gleason's Theorem and Its Applications*, Kluwer Academic Publishers, Dordrecht; Ister Science Press, Bratislava.

Dvurečenskij, A. (1993). Gleason's theorem and completeness criteria, *International Journal of Theoretical Physics* **32**, 2377–2388.

Dvurečenskij, A. (2001). A new algebraic criterion for completeness of inner product space, *Letters on Mathematical Physics* **58**, 205–208.

Dvurečenskij, A. and Pulmannová, S. (1989). A signed measure completeness criterion, *Letters on Mathematical Physics* **17**, 253–/261.

Dvurečenskij, A., Neubrunn, T., and Pulmannová, S. (1990). Finitely additive states and completeness of inner product spaces, *Foundation of Physics* **20**, 1091–1102.

Dvurečenskij, A. and Pták, P. (2002). On states on orthogonally closed subspaces of an inner product space, *Letters on Mathematical Physics* **62**, 63–70.

Eilers, M. and Horst, E. (1975). The theorem of Gleason for nonseparable Hilbert space, *International Journal of Theoretical Physics* **13**, 419–424.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space, *Journal of Mathematical Mechanics* **6**, 885–893.

Gudder, S. P. (1975). Inner product spaces, *American Mathematics Monthly* **82**, 251–252.

Gross, H. and Keller, H. A. (1977). On the definition of Hilbert space, *Manuscr. Math.* **23**, 67–90.

Hamhalter, J. (2003). *Quantum Measure Theory*, Kluwer Academic Publishers, Dordrecht.

Hamhalter, J. and Pták, P. (1987). A completeness criterion for inner product spaces, *Bulletin of the London Mathematical Society* **19**, 259–263.

Holland, Jr., S. (1969). Partial solution to Mackey's problem about modular pairs and completeness, *Canadian Journal of Mathematics* **21**, 1518–1525.

Husimi, K. (1937). Studies on the foundations of quantum mechanics 1, *Proceedings of the Physmathematical Society of Japan* **19**, 766–789.

Jajte, R. (1972). On convergence of Gleason measures, *Bull. Acad. Polon. Sci., ser. Math. Astr. Phys.* **20**, 211–214.

Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, London, New York.

Kalmbach, G. (1986). *Measures and Hilbert Lattices*, World Scientific, Singapore.

Keller, H. A. (1980). Ein nicht-klassischer Hilbertischen Raum, *Math. Z.* **172**, 41–49.

Maeda, F. and Maeda, S. (1970). *Theory of Symmetric Lattices*, Springer-Verlag, Berlin.

Pták, P. (1988). FAT–CAT (in the state space of quantum logics). In *Proceedings of "Winter School of Measure Theory*," Liptovský Ján, Czechoslovakia, pp. 113–118.

Pták, P. and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publishers, Dordrecht.

Pták, P. and Weber, H. (1998). *Two remarks on the subspace poset of an inner product space*, Contrib. Gen. Alg. 10, Proceedings of the Klagenfurt Conference, May 19–June 1, 1997, Verlag Johannes Heyn, Klagenfurt, pp. 263–267.

Pták, P. and Weber, H. (2001). Lattice properties of subspace families of inner product spaces, *Proceedings of the American Mathematical Society* **129**, 2111–2117.

Pulmannová, S. (2005). A note on homomorphisms of inner product spaces, *Demonstratio Mathematica* **38**, in press.

Sherstnev, A. N. (1974). On a notion of a charge in noncommutative scheme of measure theory, *Veroj. metod i kibern. Kazan, no.* **10–11**, 68–72 (in Russian).

Solèr, M. P. (1995). Characterization of Hilbert spaces by orthomodular spaces, *Comm. Algebra* **23**, 219–243.

Varadarajan, V. S. (1985). *Geometry of Quantum Theory*, Springer-Verlag, New York.

Wigner, E. P. (1959). *Group Theory and Its Applications to Quantum Mechanics of Atomic Spectra*, Academic Press, New York.